Eigensolution of a 2x2 Matrix

The eigenvectors and eigenvalues of a 2x2 matrix can be easily calculated in closed form. Using the standard eigensolution formula:

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$$

Where $\boldsymbol{A}$ is a square matrix, $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ and λ is the eigenvalue associated with eigenvector $\boldsymbol{v}$. For the 2x2 case, the eigensolution formula is expanded as:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The eigenvalues of the system are calculated by rearranging the equation into homogeneous form:

$$(\boldsymbol{A} - \lambda\boldsymbol{I}) \cdot \boldsymbol{v} = 0$$

or

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For this to hold true, the determinant of $\boldsymbol{A} - \lambda\boldsymbol{I}$ must be equal to zero.

$$\det(A - \lambda I) = 0$$

Solving the determinant of gives:

$$a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{12}a_{21} = 0$$

The determinant equation can then be rearranged into a polynomial of λ

$$(1)\lambda^2 + (-a_{11} - a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

The roots of the λ polynomial are then solved for using the quadratic formula.

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where:

$$a = 1$$
$$b = -a_{11} - a_{22}$$
$$c = a_{11}a_{22} - a_{12}a_{21}$$

The values of λ are the eigenvalues of the system. If the quadratic formula discriminant, $\sqrt{b^2 - 4ac}$, is positive, the matrix $\boldsymbol{A}$ will have two distinct, real roots. If the discriminant is 0, the system has 1 real root. If the discriminant is negative, the system will have two complex roots.

The eigenvalues are then plugged back into the eigensolution formula.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}$$

This equation cannot be solved for a distinct value since it reduces to a homogeneous system. Instead, the relationship of v_1 to v_2 is calculated. Expanding the matrix form into equation form:

$$a_{11}v_1 + a_{12}v_2 = \lambda v_1$$
$$a_{21}v_1 + a_{22}v_2 = \lambda v_2$$

Using the top equation, the value of v_1 is replaced with 1 (for simplicity).

$$a_{11} + a_{12}v_2 = \lambda$$

The corresponding value of v_2 is therefore:

$$v_2 = \frac{(\lambda - a_{11})}{a_{12}}$$

These values are then unitized to create the normalized eigenvector

$$v_1 = \frac{1}{\sqrt{1 + v_2^2}}$$
$$v_2 = \frac{v_2}{\sqrt{1 + v_2^2}}$$