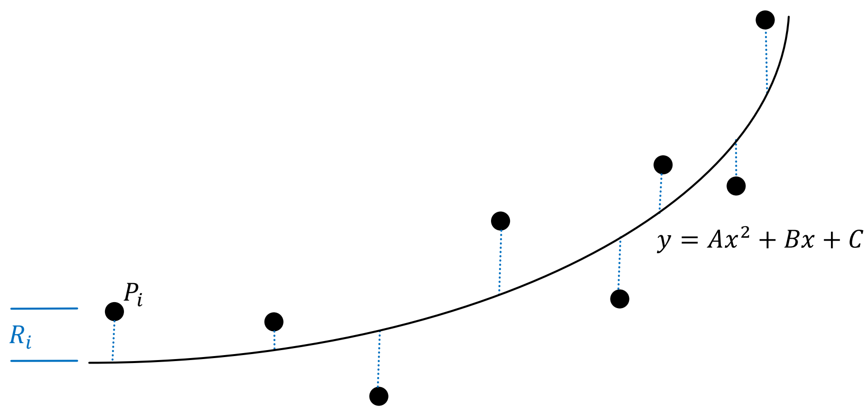
Polynomial Regression

This method fits a 2D polynomial to a set of data with $n > order$ data points. For instance, the quadratic (2nd order system) below requires at least 3 linearly independent data points to regress.



For this derivation, a quadratic equation is used, but the mathematics is expandable to any nth order system. A quadratic can be defined as:

$$y = Ax^2 + Bx + C$$

The general form of least squares regression is to calculate the coefficients to some function F that minimizes the difference between the regressed and measured data.

$$\varepsilon^2 = \sum (F(x) - y')^2$$

Where $F(x) = Ax^2 + Bx + C$ and y' are the y-components of the measured values. The equation for the least regression of a quadratic is therefore:

$$\varepsilon^2 = \sum (Ax^2 + Bx + C - y')^2$$

The minimum value of ε will be the global minimum of the regression equation. At the global minimum the derivative of ε must be 0.

$$\frac{d\varepsilon}{dy} = 0$$

For the derivative of ε to be 0, the partial derivatives of ε with respect to all the variables in the equation must also be 0. Since the x and y' values are known, we treat them as the coefficients and A, B, and C as the variables. Therefore the partial derivatives of ε with respect to A, B, and C must also be 0.

$$\frac{\partial\varepsilon}{\partial A} = 2\sum (Ax^2 + Bx + C - y')x^2 = 0$$

$$\frac{\partial\varepsilon}{\partial B} = 2\sum (Ax^2 + Bx + C - y')x = 0$$

$$\frac{\partial\varepsilon}{\partial C} = 2\sum (Ax^2 + Bx + C - y') = 0$$

Dividing by 2 on each side of the equation eliminates the 2's outside of the sum. Simplifying the equations and using the distributive law for summations, the equations can be rewritten as:

$$A\sum x^4 + B\sum x^3 + C\sum x^2 = \sum x^2 y'$$

$$A\sum x^3 + B\sum x^2 + C\sum x = \sum x y'$$

$$A\sum x^2 + B\sum x + C\sum 1 = \sum y'$$

The sum of 1 represents the number of points (n) in the regression. Converting the previous equation to matrix form gives.

$$\begin{bmatrix} \sum x^4 & \sum x^3 & \sum x^2 \\ \sum x^3 & \sum x^2 & \sum x \\ \sum x^2 & \sum x & n \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \sum x^2 y' \\ \sum x y' \\ \sum y' \end{bmatrix}$$

Solving for the polynomial coefficients A, B, and C:

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \sum x^4 & \sum x^3 & \sum x^2 \\ \sum x^3 & \sum x^2 & \sum x \\ \sum x^2 & \sum x & n \end{bmatrix}^{-1} \cdot \begin{bmatrix} \sum x^2 y' \\ \sum x y' \\ \sum y' \end{bmatrix}$$

<u>Note</u>: Calculation of the matrix inverse is outside of the scope of this section.

This method can be expanded to higher order polynomials as well. For instance, a 4th order polynomial in the form of:

$$y = Ax^4 + Bx^3 + Cx^2 + Dx + E$$

Can be solved for as:

$$\begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} \sum x^8 & \sum x^7 & \sum x^6 & \sum x^5 & \sum x^4 \\ \sum x^7 & \sum x^6 & \sum x^5 & \sum x^4 & \sum x^3 \\ \sum x^6 & \sum x^5 & \sum x^4 & \sum x^3 & \sum x^2 \\ \sum x^5 & \sum x^4 & \sum x^3 & \sum x^2 & \sum x \\ \sum x^4 & \sum x^3 & \sum x^2 & \sum x & n \end{bmatrix}^{-1} \cdot \begin{bmatrix} \sum x^4 y' \\ \sum x^3 y' \\ \sum x^2 y' \\ \sum x y' \\ \sum y' \end{bmatrix}$$

Notice that a distinct pattern arises. This same coefficient matrix can be calculated by taking the matrix product of the original polynomial with itself for each point and summing.

$$\sum \left(\begin{bmatrix} {x_i}^4 \\ {x_i}^3 \\ {x_i}^2 \\ x_i \\ 1 \end{bmatrix} \cdot \begin{bmatrix} {x_i}^4 & {x_i}^3 & {x_i}^2 & x_i & 1 \end{bmatrix} \right) = \begin{bmatrix} \sum x^8 & \sum x^7 & \sum x^6 & \sum x^5 & \sum x^4 \\ \sum x^7 & \sum x^6 & \sum x^5 & \sum x^4 & \sum x^3 \\ \sum x^6 & \sum x^5 & \sum x^4 & \sum x^3 & \sum x^2 \\ \sum x^5 & \sum x^4 & \sum x^3 & \sum x^2 & \sum x \\ \sum x^4 & \sum x^3 & \sum x^2 & \sum x & n \end{bmatrix}^{-1}$$

The solution matrix can be calculated the same way.

$$\sum\left(\begin{bmatrix} {x_i}^4 \\ {x_i}^3 \\ {x_i}^2 \\ x_i \\ 1 \end{bmatrix} \cdot y'\right) = \begin{bmatrix} \sum x^4 y' \\ \sum x^3 y' \\ \sum x^2 y' \\ \sum x y' \\ \sum y' \end{bmatrix}$$

These forms are more conducive to software generalization. By creating a power array for the x values, the powers of x can be calculated in a variable size loop that is dependent on the order of the system. In c++ pseudocode:

```
//Declare the calculation arrays
double EqnArray[Order+1];
double CoeffArray[Order+1][Order+1];
double SolnArray[Order+1];

//Calculate the polynomial variables for the current point. For the quadratic
//y = Ax^2 + Bx + C the EqnArray = [x^2 , x , 1]
EqnArray[Order] = 1;
for(int i=Order-1 ; i>=0 ; i--)
  EqnArray[i] = EqnArray[i+1]*x;

for(int n=0 ; i<NumPoints ; i++)
{
  for(int i=0 ; i<=Order ; i++)
  {
    //Accumulate points in the solution array
    SolnArray[i] += EqnArray[i]*y;

    //Accumulate points in the coeffient array
    for(int j=0 ; j<=Order ; j++)
      CoeffArray[i][j] += EqnArray[i]*EqnArray[j]*;

  }
}
```