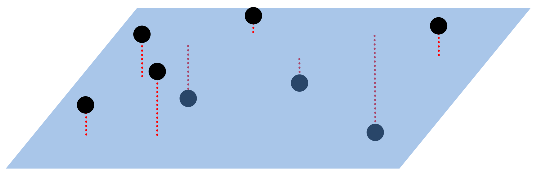
Planar/Bilinear Least Squares Regression

Planar regression calculates the best fit plane through a group of 3 or more data points. The plane is calculated by minimizing the residuals (or errors) between the plane and the original points using least squares minimization.



The least squares minimization equation is:

$$\varepsilon^2 = \sum_{i=1}^{n} (f(x_i, y_i) - z_i)^2$$

Where z_i are the observed values and $f(x_i, y_i)$ is the y-value of the surface at x_i, y_i. The equation of the plane is:

$$z = Ax + By + C$$

Plugging this value in to the regression equation gives

$$\varepsilon^2 = \sum_{i=1}^{n} (Ax_i + By_i + C - z_i)^2$$

To find the minimum residual error, the derivative of the residuals equation must be zero, which means all of the partial derivatives with respect to each coefficient must be equal to zero.

$$\frac{d\varepsilon}{dA} = \sum_{i=1}^{n} (Ax_i + By_i + C - z_i) \cdot x_i = 0$$
$$\frac{d\varepsilon}{dB} = \sum_{i=1}^{n} (Ax_i + By_i + C - z_i) \cdot y_i = 0$$
$$\frac{d\varepsilon}{dC} = \sum_{i=1}^{n} (Ax_i + By_i + C - z_i) = 0$$

These equations can be expressed in matrix form:

$$\begin{bmatrix} \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i & \sum y_i & \sum 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \sum x_i z_i \\ \sum y_i z_i \\ \sum z_i \end{bmatrix}$$

Solving for A, B, and C:

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \sum {x_i}^2 & \sum x_i y_i & \sum x_i \\ \sum x_i y_i & \sum {y_i}^2 & \sum y_i \\ \sum x_i & \sum y_i & \sum 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \sum x_i z_i \\ \sum y_i z_i \\ \sum z_i \end{bmatrix}$$