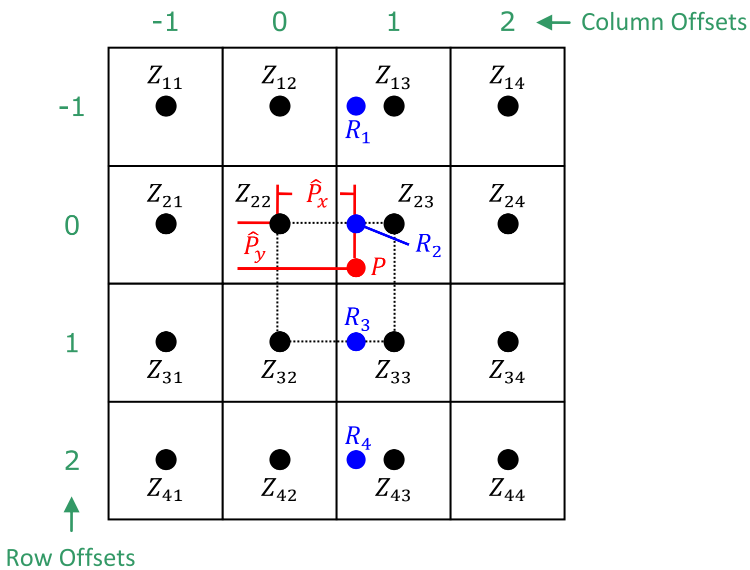
Bicubic Interpolation

Bicubic interpolation solves for the value at a new point by analyzing the 16 data points surrounding the interpolation region, see the example below.



The points Z_{22}, Z_{23}, Z_{32}, and Z_{33} are the four closest points to the interpolation point and define the interpolation region. The interpolation variables $\hat{P}_x$ and $\hat{P}_y$ are calculated by determining the normalized horizontal and vertical distance between the four closest points.

$$\hat{P}_x = \frac{P_x - X_{22}}{X_{23} - X_{22}}$$
$$\hat{P}_y = \frac{P_y - Y_{22}}{Y_{32} - Y_{22}}$$

This bicubic interpolation is for imagery, we assume a 1 pixel delta between pixels in adjacent columns and rows. Since the distance between is always 1, the values for $\hat{P}_x$ and $\hat{P}_y$ can be simplified to:

$$\hat{P}_x = P_x - \lfloor P_x \rfloor$$
$$\hat{P}_y = P_y - \lfloor P_y \rfloor$$

Where the $\lfloor\ \rfloor$ represents the floor of the value. For the horizontal interpolation portion of this algorithm, a cubic must be defined for each row of the 4x4 pixel region. These will be used to solve for the x-components of the values R_1 through R_4.

$$R_i = A_i x^3 + B_i x^2 + C_i x + D_i$$

The values of x relative to the current location are inserted into the cubic and solved for each of the 4 pixels in the row. This develops a system of linear equations that can be used to solve for the coefficients $A - D$. The solution for the first row is calculated as:

$$\begin{aligned} z_{11} &= A_1(-1)^3 + B_1(-1)^2 + C_1(-1) + D_1 \\ z_{12} &= A_1(0)^3 + B_1(0)^2 + C_1(0) + D_1 \\ z_{13} &= A_1(1)^3 + B_1(1)^2 + C_1(1) + D_1 \\ z_{14} &= A_1(2)^3 + B_1(2)^2 + C_1(2) + D_1 \end{aligned}$$

Where the value of z_{ij} is the pixel intensity. Rewriting these equations in matrix form gives:

$$[A_1 \quad B_1 \quad C_1 \quad D_1] \cdot \begin{bmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = [Z_{11} \quad Z_{12} \quad Z_{13} \quad Z_{14}]$$

Since the offsets of x are consistent for all four rows, all four rows can be solved for simultaneously as:

$$\begin{bmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{bmatrix} \cdot \begin{bmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{bmatrix}$$

For simplicity, this is rewritten in shorthand notation as:

$$[\boldsymbol{C_R}] \cdot [\boldsymbol{X}] = [\boldsymbol{Z}]$$

Where $[\boldsymbol{CR}]$ is the cubic coefficients for the four rows, $[\boldsymbol{Z}]$ is the pixel intensity values for the surrounding pixels and $[\boldsymbol{X}]$ is a constant array of offsets:

$$[\boldsymbol{X}] = \begin{bmatrix} -1^3 & 0 & 1^3 & 2^3 \\ -1^2 & 0 & 1^2 & 2^2 \\ -1^1 & 0 & 1^1 & 2^1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 8 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Since $[\boldsymbol{X}]$ is an array of constants, the inverse of $[\boldsymbol{X}]$ is an array of constants as well.

$$[\boldsymbol{X^{-1}}] = \begin{bmatrix} -{}^1/_6 & {}^1/_2 & -{}^1/_6 & 0 \\ {}^1/_2 & -1 & -{}^1/_2 & 1 \\ -{}^1/_2 & {}^1/_2 & 1 & 0 \\ {}^1/_6 & 0 & -{}^1/_6 & 0 \end{bmatrix}$$

Using $[\boldsymbol{X^{-1}}]$**,** the coefficients of the row cubics can be solved for.

$$[\boldsymbol{C_R}] = [\boldsymbol{Z}] \cdot [\boldsymbol{X^{-1}}]$$

Each of the row cubics is now solved at the horizontal normalized coordinate $\hat{P}x$:

$$\begin{aligned} A_1\hat{P}x^3 + B_1\hat{P}x^2 + C_1\hat{P}x + D_1 &= R_1 \\ A_2\hat{P}x^3 + B_2\hat{P}x^2 + C_2\hat{P}x + D_2 &= R_1 \\ A_3\hat{P}x^3 + B_3\hat{P}x^2 + C_3\hat{P}x + D_3 &= R_3 \\ A_4\hat{P}x^3 + B_4\hat{P}x^2 + C_4\hat{P}x + D_4 &= R_4 \end{aligned}$$

Converting this to matrix form

$$\begin{bmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{bmatrix} \cdot \begin{bmatrix} \hat{P}x^3 \\ \hat{P}x^2 \\ \hat{P}x \\ 1 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}$$

Written in shorthand notation:

$$[\boldsymbol{C_R}] \cdot [\boldsymbol{P_x}] = [\boldsymbol{R}]$$

To solve for the value of $\boldsymbol{P}$, a vertical cubic is fit through the row interpolation points. Using the same technique as with the row interpolations, the value of y is plugged into the cubic for each of the known row offsets and solved for the value at the associated R_i:

$$\begin{aligned} A_C y_1^3 + B_C y_1^2 + C_C y_1 + D_C &= R_1 \\ A_C y_2^3 + B_C y_2^2 + C_C y_2 + D_C &= R_1 \\ A_C y_3^3 + B_C y_3^2 + C_C y_3 + D_C &= R_3 \\ A_C y_4^3 + B_C y_4^2 + C_C y_4 + D_C &= R_4 \end{aligned}$$

Where $y_1 = -1$, $y_2 = 0$, $y_3 = 1$, and $y_4 = 2$. Converted to matrix form:

$$\begin{bmatrix} y_1^3 & y_1^2 & y_1 & 1 \\ y_2^3 & y_2^2 & y_2 & 1 \\ y_3^3 & y_3^2 & y_3 & 1 \\ y_4^3 & y_4^2 & y_4 & 1 \end{bmatrix} \cdot \begin{bmatrix} A_C \\ B_C \\ C_C \\ D_C \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}$$

In shorthand notation:

$$[\boldsymbol{Y}] \cdot [\boldsymbol{C_C}] = [\boldsymbol{R}]$$

$[\boldsymbol{Y}]$ is also an array of constant offsets:

$$[\boldsymbol{Y}] = \begin{bmatrix} -1^3 & -1^2 & -1^1 & 1 \\ 0^3 & 0^2 & 0^1 & 1 \\ 1^3 & 1^2 & 1^1 & 1 \\ 2^3 & 2^2 & 2^1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$

The inverse of $[\boldsymbol{Y}]$ is calculated as:

$$[\boldsymbol{Y}] = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/3 & -1/2 & 1 & -1/6 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The final step is to solve for the value at point P by solving the vertical cubic at the normalized vertical value $\hat{P}y$:

$$P = A_C\hat{P}y^3 + B_C\hat{P}y^2 + C_C\hat{P}y + D_C$$

Written in matrix form:

$$P = [\hat{P}y^3 \quad \hat{P}y^2 \quad \hat{P}y \quad 1] \cdot \begin{bmatrix} A_C \\ B_C \\ C_C \\ D_C \end{bmatrix}$$

Written in shorthand form:

$$P = [\boldsymbol{Py}] \cdot [\boldsymbol{C_C}]$$

All of the preceding equations can be collapsed into a single 4x4 that is only based on the known quantities: $\hat{P}x$, $\hat{P}y$, and $[\boldsymbol{Z}]$. Starting with the final equation and substituting:

$$\begin{aligned} P &= [\boldsymbol{Py}] \cdot [\boldsymbol{C_C}] && \textit{where} && [\boldsymbol{C_C}] = [\boldsymbol{Y^{-1}}] \cdot [\boldsymbol{R}] \\ P &= [\boldsymbol{Py}] \cdot [\boldsymbol{Y^{-1}}] \cdot [\boldsymbol{R}] && \textit{where} && [\boldsymbol{R}] = [\boldsymbol{C_R}] \cdot [\boldsymbol{Px}] \\ P &= [\boldsymbol{Py}] \cdot [\boldsymbol{Y^{-1}}] \cdot [\boldsymbol{C_R}] \cdot [\boldsymbol{Px}] && \textit{where} && [\boldsymbol{C_R}] = [\boldsymbol{Z}] \cdot [\boldsymbol{X^{-1}}] \end{aligned}$$

Finally reduces to:

$$P = [\boldsymbol{Py}] \cdot [\boldsymbol{Y^{-1}}] \cdot [\boldsymbol{Z}] \cdot [\boldsymbol{X^{-1}}] \cdot [\boldsymbol{Px}]$$