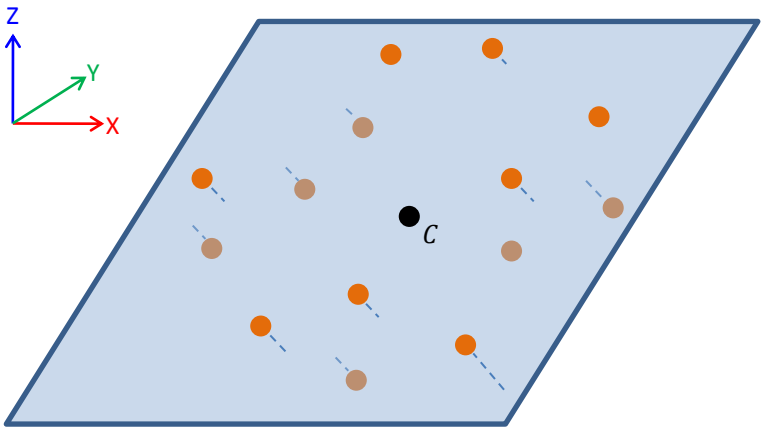
Eigensolution for Best-Fit 3D Ellipse / Best-Fit Plane

A best-fit 3D ellipse (or plane) can be found for a collection of points using the Eigensolution of the covariance matrix of the points. Take the series of points in the example below:



First the mean of the points (i.e. centroid) is calculated.

$$C_x = \frac{\sum x_i}{n}$$
$$C_y = \frac{\sum y_i}{n}$$
$$C_z = \frac{\sum z_i}{n}$$

The points are then normalized by moving them to the origin of the system (i.e. subtracting the centroid from each point).

$$x_i' = x_i - C_x$$
$$y_i' = y_i - C_y$$
$$z_i' = z_i - C_z$$

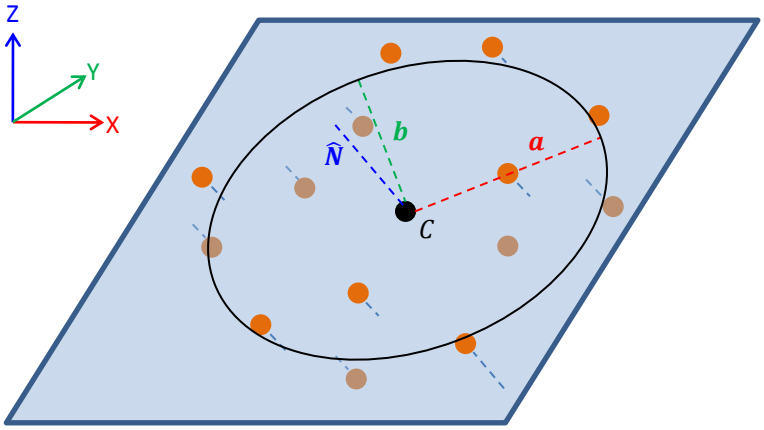
The covariance is then calculated as:

$$A = \begin{bmatrix} \sum x_i'^2 & \sum x_i'y_i' & \sum x_i'z_i' \\ \sum x_i'y_i' & \sum y_i'^2 & \sum y_i'z_i' \\ \sum x_i'z_i' & \sum y_i'z_i' & \sum z_i'^2 \end{bmatrix}$$

The eigenvectors and values of this symmetric matrix are then calculated. *Note: The eigensolution of a 3x3 matrix is detailed in a separate section here.*

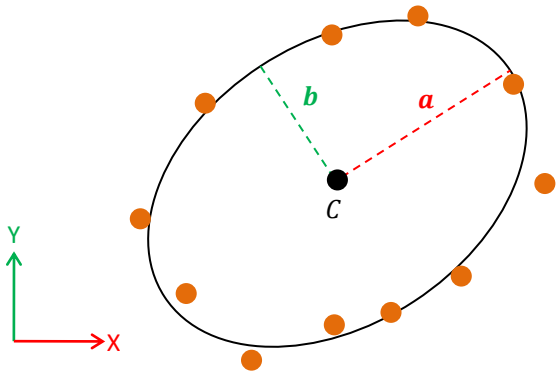
$$\begin{bmatrix} \boldsymbol{V}_1 & \boldsymbol{V}_2 & \boldsymbol{V}_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} = \text{SolveEigen3x3}(A)$$

The eigenvectors $(\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3)$ of the matrix A represent orthogonal axes of the best-fit ellipse. The eigenvector associated with the smallest eigenvalue represents the normal direction $(\hat{N})$ of the plane. The eigenvector associated with the largest eigenvalue represents a scalar multiple of the major axis (a) of an ellipse on the plane and the mid value represents a scalar multiple of the minor axis (b) of the ellipse.



the vector directions of the major and minor axes of the ellipse. The associated eigenvalues (λ_1 , λ_2) are scalar multiples of the major and minor axis lengths. The eigenvectors must first be unitized.

$$\hat{\boldsymbol{V}}_1 = \frac{\boldsymbol{V}_1}{|\boldsymbol{V}_1|} \text{ and } \hat{\boldsymbol{V}}_2 = \frac{\boldsymbol{V}_2}{|\boldsymbol{V}_2|}$$



The major axis ($\boldsymbol{a}$) will be defined by the largest eigenvalue and its associated eigenvector. The minor axis ($\boldsymbol{b}$) will be defined by the other.

$$\begin{aligned}
&\text{if}(\lambda_1 > \lambda_2)\\
&\quad \hat{\boldsymbol{V}}_a = \hat{\boldsymbol{V}}_1\\
&\quad \hat{\boldsymbol{V}}_b = \hat{\boldsymbol{V}}_2\\
&\quad \lambda_a = \lambda_1\\
&\quad \lambda_b = \lambda_2\\
&\text{else}\\
&\quad \hat{\boldsymbol{V}}_a = \hat{\boldsymbol{V}}_2\\
&\quad \hat{\boldsymbol{V}}_b = \hat{\boldsymbol{V}}_1\\
&\quad \lambda_a = \lambda_2\\
&\quad \lambda_b = \lambda_1
\end{aligned}$$

The eigenvalues are a scaled representation of the major and minor axes, not the lengths themselves. To calculate the semi-major and semi-minor axis lengths, we need to normalize by the number of points used.

$$|a| = 4 \cdot \sqrt{\lambda_a / N}$$

$$|b| = 4 \cdot \sqrt{\lambda_b / N}$$

The final major and minor axis vectors are calculated by scaling the unitized eigenvectors by the axis lengths.

$$\boldsymbol{a} = \widehat{\boldsymbol{V}}_a \cdot |a|$$
$$\boldsymbol{b} = \widehat{\boldsymbol{V}}_b \cdot |b|$$