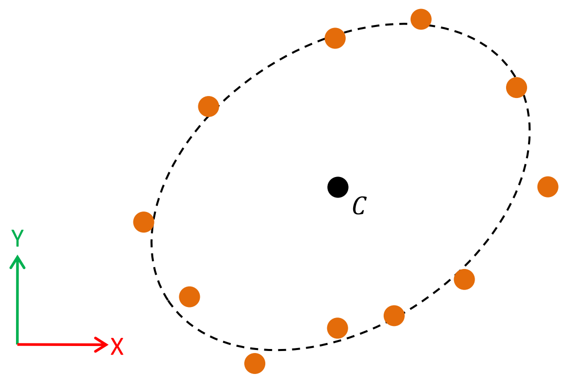
Eigensolution for Best Fit Ellipse

A best-fit ellipse can be found for a collection of points using the Eigensolution of the covariance matrix of the points. Take the series of points in the example below:



First the mean of the points (i.e. centroid) is calculated.

$$x_C = \frac{\sum x_i}{n}$$
$$y_C = \frac{\sum y_i}{n}$$

The points are then normalized by moving them to the origin of the system (i.e. subtracting the centroid from each point).

$$x_i' = x_i - x_C$$
$$y_i' = y_i - y_C$$

The covariance is then calculated as:

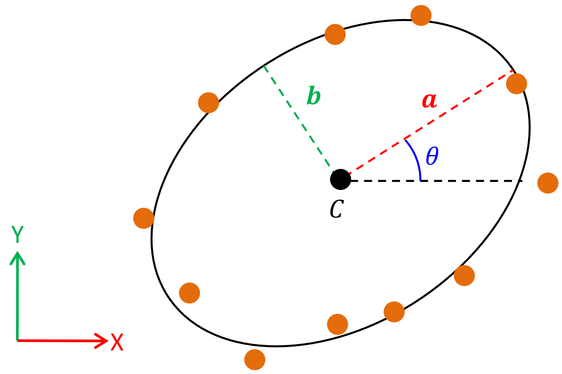
$$A = \begin{bmatrix} \sum x_i'^2 & \sum x_i' y_i' \\ \sum x_i' y_i' & \sum y_i'^2 \end{bmatrix}$$

The eigenvectors and values of this symmetric matrix are then calculated. *<u>Note</u>: The eigensolution of a 2x2 matrix is detailed in a separate section <u>here</u>.*

$$[\boldsymbol{V}_1, \boldsymbol{V}_2, \lambda_1, \lambda_2] = \text{SolveEigen2x2}(A)$$

The eigenvectors $(\boldsymbol{V}_1 , \boldsymbol{V}_2)$ of the matrix A represent the vector directions of the major and minor axes of the ellipse. The associated eigenvalues (λ_1 , λ_2) are scalar multiples of the major and minor axis lengths. The eigenvectors must first be unitized.

$$\widehat{\boldsymbol{V}}_1 = \frac{\boldsymbol{V}_1}{|\boldsymbol{V}_1|} \text{ and } \widehat{\boldsymbol{V}}_2 = \frac{\boldsymbol{V}_2}{|\boldsymbol{V}_2|}$$



The major axis $(\boldsymbol{a})$ will be defined by the largest eigenvalue and its associated eigenvector. The minor axis $(\boldsymbol{b})$ will be defined by the other.

$$
\begin{aligned}
&\text{if}(\lambda_1 > \lambda_2) \\
&\quad \widehat{\boldsymbol{V}}_a = \widehat{\boldsymbol{V}}_1 \\
&\quad \widehat{\boldsymbol{V}}_b = \widehat{\boldsymbol{V}}_2 \\
&\quad \lambda_a = \lambda_1 \\
&\quad \lambda_b = \lambda_2 \\
&\text{else} \\
&\quad \widehat{\boldsymbol{V}}_a = \widehat{\boldsymbol{V}}_2 \\
&\quad \widehat{\boldsymbol{V}}_b = \widehat{\boldsymbol{V}}_1 \\
&\quad \lambda_a = \lambda_2 \\
&\quad \lambda_b = \lambda_1
\end{aligned}
$$

The eigenvalues are a scaled representation of the major and minor axes, not the lengths themselves. To calculate the semi-major and semi-minor axis lengths, we need to normalize by the number of points used.

$$
a = |\boldsymbol{a}| = 4 \cdot \sqrt{\lambda_a / N}
$$
$$
b = |\boldsymbol{b}| = 4 \cdot \sqrt{\lambda_b / N}
$$

The final major and minor axis vectors are calculated by scaling the unitized eigenvectors by the axis lengths.

$$
\boldsymbol{a} = \widehat{\boldsymbol{V}}_a \cdot a
$$
$$
\boldsymbol{b} = \widehat{\boldsymbol{V}}_b \cdot b
$$

The ellipse can also be described parametrically as:

$$
x = x_C + a \cdot \cos(2\pi t) \cdot \cos(\theta) - b \cdot \sin(2\pi t) \cdot \sin(\theta)
$$
$$
y = y_C + a \cdot \cos(2\pi t) \cdot \sin(\theta) + b \cdot \sin(2\pi t) \cdot \cos(\theta)
$$

Where t is the independent parameter and θ is the rotational angle of the major axis from the x-axis.