Rotation & Translation Matrices

Rotation and translation matrices allow points to easily be moved around in a coordinate system, or moved from one coordinate system to another. All the following transformation matrices are in 3D homogenous coordinate format.

Translation

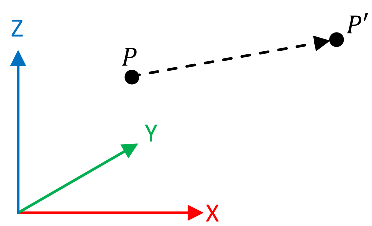
The translation matrix translates a point in 3D Cartesian space to another location specified by an offset vector $(dx\,, dy\,, dz)$.

$$\begin{aligned} x' &= x + dx \\ y' &= y + dy \\ z' &= z + dz \end{aligned}$$

Converting these equations to matrix form gives:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

A fourth coordinate is added to the matrix to make the system homogeneous. This final parameter, typically depicted as w, is typically 1 unless the point is reaching infinity. The translation matrix T is the operator that translates the point from its original position to its new position.



$$T = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A point P is translated to position P' converting the point P to a homogeneous representation and multiplying by the rotation matrix.

Rotation About X, Y, and Z-Axes

The x-axis rotation matrix rotates an object in 3D Cartesian space around the x-axis of the coordinate system by an angle θ_X. The rotation equations are derived from a general definition of a rotation in 2 degrees:

$$\begin{aligned} x' &= x \cdot \cos(\theta) - y \cdot \sin(\theta) \\ y' &= x \cdot \sin(\theta) + y \cdot \cos(\theta) \end{aligned}$$

Applying this rotation logic to the y and z axes gives the rotation about the x-axis.

$$\begin{aligned} x' &= x \\ y' &= y \cdot \cos(\theta_X) - z \cdot \sin(\theta_X) \\ z' &= y \cdot \sin(\theta_X) + z \cdot \cos(\theta_X) \end{aligned}$$

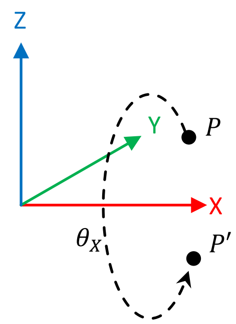
Putting these equations into matrix form gives:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_X) & -\sin(\theta_X) & 0 \\ 0 & \sin(\theta_X) & \cos(\theta_X) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

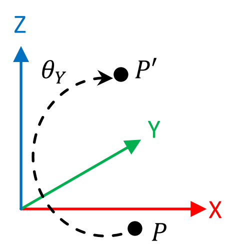
Yielding the rotation matrix for rotating about the x-axis.

$$R_X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_X) & -\sin(\theta_X) & 0 \\ 0 & \sin(\theta_X) & \cos(\theta_X) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

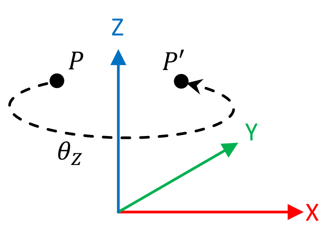
The rotation matrices for rotating about the y and z-axes are formulated in the same fashion but applying the rotation to the other axes. The basic rotation matrices for all three axes are shown below.



$$R_X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_X) & -\sin(\theta_X) & 0 \\ 0 & \sin(\theta_X) & \cos(\theta_X) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$R_Y = \begin{bmatrix} \cos(\theta_Y) & 0 & \sin(\theta_Y) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta_Y) & 0 & \cos(\theta_Y) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

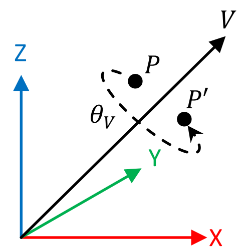


$$R_Z = \begin{bmatrix} \cos(\theta_Z) & -\sin(\theta_Z) & 0 & 0 \\ \sin(\theta_Z) & \cos(\theta_Z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation About Vector

This rotation matrix rotates a point in 3D Cartesian space around a vector (V) originating at the origin by an angle θ_V.

[TBD: Add mathematical basis]



A point P is rotated about the vector V to position P' by converting the point P to its homogeneous representation and multiplying by the rotation matrix.

$$R_V = \begin{bmatrix} 1 + (x^2 - 1) \cdot C & -zS + xyC & yS + xzC & 0 \\ zS + xyC & 1 + (y^2 - 1) \cdot C & -xS + yzC & 0 \\ -yS + xzC & xS + yzC & 1 + (z^2 - 1) \cdot C & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} V \equiv (x, y, z) \\ S = \sin(\theta_V) \\ C = (1 - \cos(\theta_V)) \end{array}$$