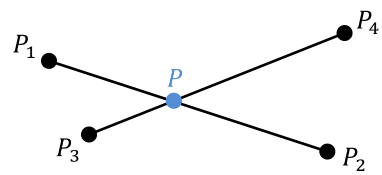
Parametric Line Intersection

Finds the point of intersection between two 2D parametric lines.



Using the parametric line equations:

$$x = x_0 + \alpha \cdot t$$
$$y = y_0 + \beta \cdot t$$

Setting the parametric equations in terms of the lines above:

$$x = x_1 + (x_2 - x_1) \cdot s \qquad \text{and} \qquad x = x_3 + (x_4 - x_3) \cdot t$$
$$y = y_1 + (y_2 - y_1) \cdot s \qquad \qquad \qquad y = y_3 + (y_4 - y_3) \cdot t$$

The two pairs of equations can be converted to a linear system of equations by setting the two x equations equal and setting the two y equations equal.

$$x_1 + (x_2 - x_1) \cdot s = x_3 + (x_4 - x_3) \cdot t$$
$$y_1 + (y_2 - y_1) \cdot s = y_3 + (y_4 - y_3) \cdot t$$

The equations can be rearranged to solve in terms of s and t.

$$x_{21} \cdot s - x_{43} \cdot t = x_{31}$$
$$y_{21} \cdot s - y_{43} \cdot t = y_{31}$$

Where:

$$x_{21} = x_2 - x_1$$
$$y_{21} = y_2 - y_1$$
$$x_{31} = x_3 - x_1$$
$$\ldots$$

These equations can now be converted to matrix form.

$$\begin{bmatrix} x_{21} & -x_{43} \\ y_{21} & -y_{43} \end{bmatrix} \cdot \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} x_{31} \\ y_{31} \end{bmatrix}$$

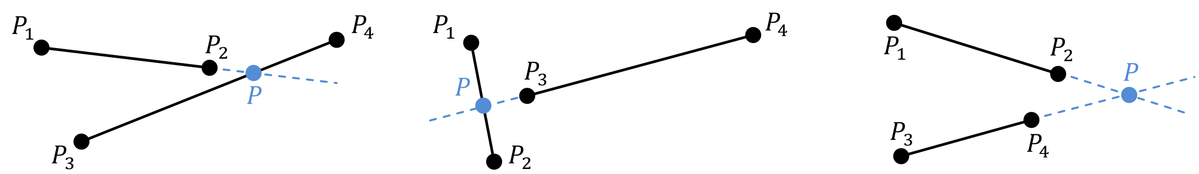
And solved for s and t.

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} x_{21} & -x_{43} \\ y_{21} & -y_{43} \end{bmatrix}^{-1} \cdot \begin{bmatrix} x_{31} \\ y_{31} \end{bmatrix}$$
$$\begin{bmatrix} s \\ t \end{bmatrix} = \frac{1}{x_{43}y_{21} - x_{21}y_{43}} \cdot \begin{bmatrix} -y_{43} & x_{43} \\ -y_{21} & x_{21} \end{bmatrix} \cdot \begin{bmatrix} x_{31} \\ y_{31} \end{bmatrix}$$

$$s = \frac{x_{43}y_{31} - x_{31}y_{43}}{x_{43}y_{21} - x_{21}y_{43}} \qquad \text{and} \qquad t = \frac{x_{21}y_{31} - x_{31}y_{21}}{x_{43}y_{21} - x_{21}y_{43}}$$

The values of s and t give the percent distance that the intersection is between the endpoints on each line. If the values of s and t are between 0 and 1, then the intersection point lies internal to the two line segments. If s or t are greater than 1 or less than 0, the lines intersect but at some point external point. If the lines are parallel the denominator of both equations will become 0 and cause the value of s and t to become infinite. This is as expected since parallel lines are defined as meeting at infinity.

The physical interpretation of s and t are shown below. The left hand image shows a case where $s > 1$ and $0 \leq t \leq 1$. The point is internal in the second line, but off of the first line. The middle shows the opposite. In this case $0 \leq s \leq 1$ and $t < 0$. The point is internal in the first line but off of the second. The right figure shows the most common case, where the intersection is outside of both lines. In this figure $s > 1$ and $t > 1$.



The s and t values can then be used to calculate the location of the intersection point. Using s or t, plug back into the original parametric line equation and solve for x and y.

$$x = x_1 + (x_2 - x_1) \cdot s$$
$$y = y_1 + (y_2 - y_1) \cdot s$$

This method, while more laborious than some other Cartesian intersection calculations, is useful in that it automatically solves for the location of the intersection point on the lines.