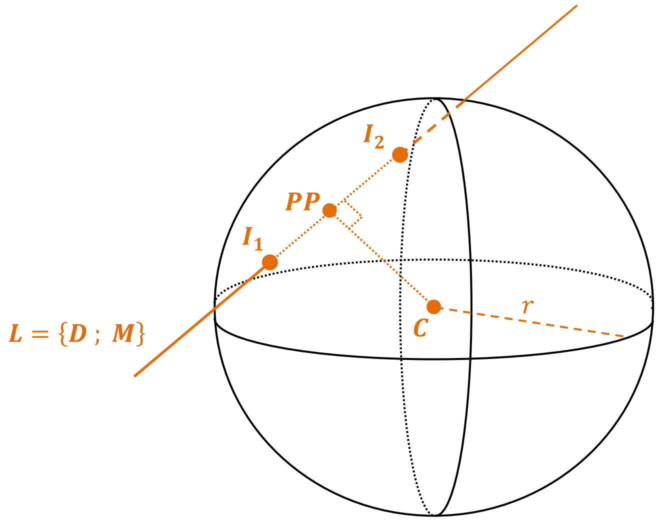
Intersection of Plücker Line and Sphere

A Plücker line and a sphere can intersect at 0, 1, or 2 points.



Using a Plücker line in the form of:

$$L = \{D_x\ , D_y\ , D_z\ ; M_x\ , M_y\ , M_z\}$$

and a sphere in the form of:

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2$$

The intersection points I_1 and I_2 of the line and the sphere can be solved for. First, the principal point of the line must be defined. This is the point of closest approach on the line to the origin. For a Plücker line, the principal point (PP) is defined as the cross product of the direction vector and the moment vector of the line.

$$PP_x = D_y M_z - D_z M_y$$
$$PP_y = D_z M_x - D_x M_z$$
$$PP_z = D_x M_y - D_y M_x$$

Starting from the principal point, the intersection points must line along the direction vector of the line.

$$\boldsymbol{I} = \boldsymbol{PP} + d\boldsymbol{D}$$

In component form:

$$I_x = PP_x + dD_x$$
$$I_y = PP_y + dD_y$$
$$I_z = PP_y + dD_z$$

Plugging the components of $\boldsymbol{I}$ into the sphere equation, you get:

$$(PP_x + dD_x - x_c)^2 + \left(PP_y + dD_y - y_c\right)^2 + \left(PP_y + dD_z - z_c\right)^2 = r^2$$

Expanding the algebra:

$$PP_x^2 + 2(PP_xD_x)d - PP_xx_c + (D_x^2)d^2 - 2(D_xx_c)d - PP_xx_c - x_c^2$$
$$PP_y^2 + 2(PP_yD_y)d - PP_yy_c + (D_y^2)d^2 - 2(D_yy_c)d - PP_yy_c - y_c^2$$
$$PP_z^2 + 2(PP_zD_z)d - PP_zz_c + (D_z^2)d^2 - 2(D_zz_c)d - PP_zz_c - z_c^2 = r^2$$

And reorganizing it in terms of d:

$$\left[D_x^2 + D_y^2 + D_z^2\right] d^2 +$$
$$2\left[D_xPP_x + D_yPP_y + D_zPP_z - D_xx_c - D_yy_c - D_zz_c\right] d +$$
$$\left[PP_x^2 - 2PP_xx_c - x_c^2 + PP_y^2 - 2PP_yy_c - y_c^2 + PP_z^2 - 2PP_zz_c - z_c^2\right] = r^2$$

Where each term represents a coefficient of the quadratic equation:

$$at^2 + bt + c = 0$$

Where:

$$a = D_x^2 + D_y^2 + D_z^2$$
$$b = 2\left[D_xPP_x + D_yPP_y + D_zPP_z - D_xx_c - D_yy_c - D_zz_c\right]$$
$$c = PP_x^2 - 2PP_xx_c - x_c^2 + PP_y^2 - 2PP_yy_c - y_c^2 + PP_z^2 - 2PP_zz_c - z_c^2 - r^2$$

Expanding the principal point definition and reanalyzing the coefficients as vectors allows for additional simplifications:

$$\begin{aligned}
a &= \boldsymbol{D} \circ \boldsymbol{D} \\
b &= 2[\boldsymbol{D} \circ \boldsymbol{PP} - \boldsymbol{D} \circ \boldsymbol{C}] \\
&= 2[\boldsymbol{D} \circ (\boldsymbol{D} \times \boldsymbol{M}) - \boldsymbol{D} \circ \boldsymbol{C}] \\
&= -2\boldsymbol{D} \circ \boldsymbol{C} \\
c &= (PP_x - x_c)^2 + \left(PP_y - y_c\right)^2 + (PP_z - z_c)^2 - r^2 \\
&= (\boldsymbol{PP} - \boldsymbol{C})^2 - r^2
\end{aligned}$$

Expansion of term b, shows that the dot of $\boldsymbol{D}$ and $\boldsymbol{PP}$ is actually a vector triple product containing $\boldsymbol{D}$ twice. This always equals zero so the term can be removed. Additionally, term c can be reduced to a difference and a dot squared.

Using a, b, and c, the quadratic formula is used to solve for a distance d:

$$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The value(s) of d represent the distance along the Plücker line, starting at the principal point, of the intersection points. If both roots of d are real (i.e. the discriminant is positive), the line intersects the sphere and the two values of d can be used to solve for points I_1 and I_2. If the roots are the same (i.e. the discriminant is 0) the line is tangential to the sphere and only intersects at the one point. If both roots are complex (i.e. the discriminant is negative) the line does not intersect the sphere.

$$I_{xi} = PP_x + D_xd_i$$
$$I_{yi} = PP_y + D_yd_i$$
$$I_{zi} = PP_z + D_zd_i$$